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A Bäcklund transformation and nonlinear superposition formula for the Belov–Chaltikian lattice

Xing-Biao Hu†‡ and Zuo-Nong Zhu§

† CCAST (World Laboratory), PO Box 8730, Beijing 100080, People's Republic of China
‡ State Key Laboratory of Scientific and Engineering Computing, Institute of Computational Mathematics and Scientific Engineering Computing, Academia Sinica, PO Box 2719, Beijing 100080, People's Republic of China
§ Department of Mathematics, Hong Kong Baptist University, Kowloon Tong, Hong Kong, People's Republic of China

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Abstract. The so-called Belov–Chaltikian lattice is considered. By the dependent variable transformation, the Belov–Chaltikian lattice is transformed into a trilinear form. By introducing an auxiliary variable, we further transform it into the bilinear form. A corresponding Bäcklund transformation for it is obtained. Furthermore, a nonlinear superposition formula is proved rigorously. As an application of the obtained results, soliton solutions are derived.

1. Introduction

The so-called Belov–Chaltikian (BC) lattice is [1]

$$\begin{aligned} \dot{b_n} &= b_n(b_{n+1} - b_{n-1}) - c_n + c_{n-1} \\ \dot{c_n} &= c_n(b_{n+2} - b_{n-1}) \end{aligned} \tag{1}$$

which was found in the study of lattice analogues of W-algebras. This system may be viewed as an extension of the Lotka–Volterra lattice (which appears as a $c_n = 0$ reduction of the above system). Some research on the BC lattice has been conducted. For example, Belov and Chaltikian established the bi-Hamiltonian structure of this system [1]. Integrable discretization for the BC lattice and related results were obtained by Suris [2]. In [3], Hikami and Inoue also considered the system. The purpose of this paper is to derive a Bäcklund transformation (BT) for the BC lattice and establish a nonlinear superposition formula for it. As a result, multisoliton solutions are derived step by step.

We now consider the system (1). By the dependent variable transformation

$$b_n = \left(\ln \frac{f_{n+\frac{1}{2}}}{f_{n-\frac{1}{2}}}\right)_t \qquad c_n = \frac{f_{n+\frac{5}{2}} f_{n-\frac{3}{2}}}{f_{n+\frac{3}{2}} f_{n-\frac{1}{2}}}$$

(1) can be transformed into the trilinear form:

$$\begin{aligned} f_{n+\frac{1}{2}\,tt} f_{n+\frac{3}{2}} f_{n-\frac{1}{2}} &= -f_{n+\frac{5}{2}} f_{n-\frac{3}{2}} f_{n+\frac{1}{2}} + f_{n-\frac{1}{2}} f_{n+\frac{1}{2}\,t} f_{n+\frac{3}{2}\,t} \\ &\quad - f_{n+\frac{1}{2}} f_{n-\frac{1}{2}\,t} f_{n+\frac{3}{2}\,t} + f_{n+\frac{3}{2}} f_{n-\frac{1}{2}\,t} f_{n+\frac{1}{2}\,t} + f_{n+\frac{1}{2}} f_{n-\frac{1}{2}} f_{n+\frac{3}{2}}. \end{aligned} \tag{2}$$

We can rewrite (2) as

$$\begin{aligned} & e^{D_n}(D_t^2 e^{\frac{1}{2}D_n} f_n \cdot f_n) \cdot (e^{\frac{1}{2}D_n} f_n \cdot f_n) + (D_t^2 e^{\frac{1}{2}D_n} f_n \cdot f_n)(e^{\frac{3}{2}D_n} f_n \cdot f_n) \\ & \quad - e^{\frac{1}{2}D_n}(D_t^2 e^{D_n} f_n \cdot f_n) \cdot (e^{D_n} f_n \cdot f_n) \\ & \quad = -2e^{\frac{3}{2}D_n}(e^{D_n} f_n \cdot f_n) \cdot f_n^2 + 2(e^{\frac{1}{2}D_n} f_n \cdot f_n)(e^{\frac{3}{2}D_n} f_n \cdot f_n) \end{aligned} \tag{3}$$

where the bilinear operators are defined as follows [4–6]

$$D_x^m D_t^n a \cdot b \equiv \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\right)^m \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'}\right)^n a(x,t)b(x',t')|_{x'=x,t'=t}$$

$$\exp(\delta D_n) a_n \cdot b_n \equiv \exp\left[\delta\left(\frac{\partial}{\partial n} - \frac{\partial}{\partial n'}\right)\right] a(n)b(n')|_{n'=n} = a(n+\delta)b(n-\delta).$$

By introducing an auxiliary variable z and using (A.1), (A.2), we can decouple (3) into the following bilinear form

$$(D_t^2 e^{\frac{1}{2}D_n} - D_z e^{\frac{1}{2}D_n}) f_n \cdot f_n = 0 \tag{4}$$

$$(D_z e^{D_n} - D_t^2 e^{D_n} + 2e^{2D_n} - 2e^{D_n}) f_n \cdot f_n = 0. \tag{5}$$

It is remarked that in the continous case the technique to decouple multilinear equations into bilinear ones by introducing auxiliary variables was first proposed by Hirota and Satsuma. They applied it to the Lax fifth-order KdV equation and a model equation for shallow water waves [7, 4].

This paper is organized as follows. In section 2 we give a BT for equations (4) and (5). Then in section 3, we give a brief proof of a nonlinear superposition formula. Some particular solutions of equations (4) and (5) are then found through this formula. Finally, section 4 summarizes the obtained results. The appendix lists some bilinear operator identities used in this paper.

2. A BT for the BC lattice

In this section, we derive a BT for equations (4) and (5). The result obtained is as follows.

Proposition 1. A BT for equations (4) and (5) is

$$(D_t + \lambda^{-1} e^{-D_n} + \mu) f_n \cdot g_n = 0 \tag{6}$$

$$D_z f_n \cdot g_n = [\lambda^{-1} D_t e^{-D_n} + \lambda^{-1}\mu e^{-D_n} + \gamma] f_n \cdot g_n \tag{7}$$

$$[e^{\frac{1}{2}D_n} D_t + \mu e^{\frac{1}{2}D_n} - \lambda e^{\frac{3}{2}D_n} + \omega e^{-\frac{3}{2}D_n}] f_n \cdot g_n = 0 \tag{8}$$

where λ, μ, γ and ω are arbitrary constants.

Proof. Let $f(n)$ be a solution of equations (4) and (5). If we can show that equations (6)–(8) guarantee that the following two relations

$$P_1 \equiv (D_t^2 e^{\frac{1}{2}D_n} - D_z e^{\frac{1}{2}D_n}) g_n \cdot g_n = 0$$

$$P_2 \equiv (D_z e^{D_n} - D_t^2 e^{D_n} + 2e^{2D_n} - 2e^{D_n}) g_n \cdot g_n = 0$$

hold, then equations (6)–(8) form a BT.

By using (A3)–(A7) and (6), (7), we have

$$
\begin{aligned}
-[\mathrm{e}^{\frac{1}{2}D_n} f_n \cdot f_n]P_1 &= 2D_t \cosh(\tfrac{1}{2}D_n)[D_t f_n \cdot g_n] \cdot f_n g_n - 2\sinh(\tfrac{1}{2}D_n)(D_z f_n \cdot g_n) \cdot f_n g_n \\
&= -2\lambda^{-1} D_t \cosh(\tfrac{1}{2}D_n)[\mathrm{e}^{-D_n} f_n \cdot g_n] \cdot f_n g_n - 2\sinh(\tfrac{1}{2}D_n)(D_z f_n \cdot g_n) \cdot f_n g_n \\
&= 2\lambda^{-1} \sinh(\tfrac{1}{2}D_n)\{[D_t \mathrm{e}^{-D_n} f_n \cdot g_n] \cdot f_n g_n + [D_t f_n \cdot g_n] \cdot [\mathrm{e}^{-D_n} f_n \cdot g_n]\} \\
&\quad -2\sinh(\tfrac{1}{2}D_n)(D_z f_n \cdot g_n) \cdot f_n g_n \\
&= 2\lambda^{-1} \sinh(\tfrac{1}{2}D_n)[D_t \mathrm{e}^{-D_n} f_n \cdot g_n] \cdot f_n g_n \\
&\quad +2\lambda^{-1}\mu \sinh(\tfrac{1}{2}D_n)[\mathrm{e}^{-D_n} f_n \cdot g_n] \cdot f_n g_n - 2\sinh(\tfrac{1}{2}D_n)(D_z f_n \cdot g_n) \cdot f_n g_n \\
&= 0.
\end{aligned}
$$

On the other hand, by using (A3)–(A10) and (6)–(8), we have

$$
\begin{aligned}
-[\mathrm{e}^{D_n} f_n \cdot f_n]P_2 &= 2\sinh(D_n)(D_z f_n \cdot g_n) \cdot f_n g_n - 2D_t \cosh(D_n)[D_t f_n \cdot g_n] \cdot f_n g_n \\
&\quad +4\sinh(\tfrac{1}{2}D_n)[\mathrm{e}^{\frac{3}{2}D_n} f_n \cdot g_n] \cdot [\mathrm{e}^{-\frac{3}{2}D_n} f_n \cdot g_n] \\
&= 2\sinh(D_n)[(\lambda^{-1} D_t \mathrm{e}^{-D_n} + \lambda^{-1}\mu \mathrm{e}^{-D_n}) f_n \cdot g_n] \cdot f_n g_n \\
&\quad +2D_t \cosh(D_n)[\lambda^{-1}\mathrm{e}^{-D_n} f_n \cdot g_n] \cdot f_n g_n \\
&\quad +4\sinh(\tfrac{1}{2}D_n)[\mathrm{e}^{\frac{3}{2}D_n} f_n \cdot g_n] \cdot [\mathrm{e}^{-\frac{3}{2}D_n} f_n \cdot g_n] \\
&= 2\lambda^{-1}\{2\sinh(\tfrac{1}{2}D_n)[\mathrm{e}^{-\frac{3}{2}D_n} f_n \cdot g_n] \cdot [D_t \mathrm{e}^{\frac{1}{2}D_n} f_n \cdot g_n] \\
&\quad -\sinh(D_n)[\mathrm{e}^{-D_n} f_n \cdot g_n] \cdot [D_t f_n \cdot g_n]\} + 2\lambda^{-1}\mu \sinh(D_n)[\mathrm{e}^{-D_n} f_n \cdot g_n] \cdot f_n g_n \\
&\quad +4\sinh(\tfrac{1}{2}D_n)[\mathrm{e}^{\frac{3}{2}D_n} f_n \cdot g_n] \cdot [\mathrm{e}^{-\frac{3}{2}D_n} f_n \cdot g_n] \\
&= 4\lambda^{-1} \sinh(\tfrac{1}{2}D_n)[\mathrm{e}^{-\frac{3}{2}D_n} f_n \cdot g_n] \cdot [D_t \mathrm{e}^{\frac{1}{2}D_n} f_n \cdot g_n] \\
&\quad +4\lambda^{-1}\mu \sinh(D_n)[\mathrm{e}^{-D_n} f_n \cdot g_n] \cdot f_n g_n \\
&\quad +4\sinh(\tfrac{1}{2}D_n)['e^{\frac{3}{2}D_n} f_n \cdot g_n] \cdot [\mathrm{e}^{-\frac{3}{2}D_n} f_n \cdot g_n] \\
&= 4\lambda^{-1} \sinh(\tfrac{1}{2}D_n)[\mathrm{e}^{-\frac{3}{2}D_n} f_n \cdot g_n] \cdot [D_t \mathrm{e}^{\frac{1}{2}D_n} f_n \cdot g_n] \\
&\quad +4\lambda^{-1}\mu \sinh(\tfrac{1}{2}D_n)[\mathrm{e}^{-\frac{3}{2}D_n} f_n \cdot g_n] \cdot [\mathrm{e}^{\frac{1}{2}D_n} f_n \cdot g_n] \\
&\quad +4\sinh(\tfrac{1}{2}D_n)[\mathrm{e}^{\frac{3}{2}D_n} f_n \cdot g_n] \cdot [\mathrm{e}^{-\frac{3}{2}D_n} f_n \cdot g_n] \\
&= 0.
\end{aligned}
$$

Thus we have completed the proof of proposition 1. □

By using (6)–(8), we can easily find the following solutions from the trivial solution $f_n = 1$:

$$g_n = 1 + \exp(\eta)$$

with

$$\lambda = \pm\sqrt{\frac{\mathrm{e}^{\frac{1}{2}p} - \mathrm{e}^{\frac{3}{2}p}}{\mathrm{e}^{\frac{3}{2}p} - \mathrm{e}^{-\frac{3}{2}p}}} \qquad \mu = -\lambda^{-1} \qquad \gamma = \lambda^{-2} \qquad \omega = \lambda + \lambda^{-1}$$

where $\eta = pn + qz + rt + \eta^0$, $q = \lambda^{-2}(\mathrm{e}^{2p} - 1)$, $r = \lambda^{-1}(\mathrm{e}^p - 1)$.

3. A nonlinear superposition formula

In the following, we shall simply denote, without confusion, $f_n(t) = f(n, t) = f(n)$ or f. The result reached is as follows.

Proposition 2. Let f_0 be a solution of equations (4) and (5) and suppose that f_i $(i=1,2)$ are solutions of (4), (5) which are related to f_0 under the BT (6)–(8) with parameters $(\lambda_i,\mu_i,\gamma_i,\omega_i)$, i.e. $f_0 \overset{(\lambda_i,\mu_i,\gamma_i,\omega_i)}{\longrightarrow} f_i$ $(i=1,2)$, $\lambda_1\lambda_2\neq 0$, $f_j\neq 0$ $(j=0,1,2)$. Then f_{12} defined by

$$\exp(-\tfrac{1}{2}D_n)f_0\cdot f_{12} = c[\lambda_1\exp(-\tfrac{1}{2}D_n)-\lambda_2\exp(\tfrac{1}{2}D_n)]f_1\cdot f_2 \qquad (c \text{ is a nonzero constant}) \tag{9}$$

is a new solution which is related to f_1 and f_2 under the BT (6)–(8) with parameters $(\lambda_2,\mu_2,\gamma_2,\omega_2)$, $(\lambda_1,\mu_1,\gamma_1,\omega_1)$ respectively.

Proof. Similar to the deduction in [8], we can show that

$$(D_t+\lambda_2^{-1}\mathrm{e}^{-D_n}+\mu_2)f_1\cdot f_{12}=0$$
$$(D_t+\lambda_1^{-1}\mathrm{e}^{-D_n}+\mu_1)f_2\cdot f_{12}=0$$

and

$$-D_t f_1\cdot f_2+(\mu_1-\mu_2)f_1f_2-\frac{1}{c\lambda_1\lambda_2}\mathrm{e}^{-D_n}f_0\cdot f_{12}=0 \tag{10}$$

$$\frac{1}{2c}D_t\mathrm{e}^{-\frac{1}{2}D_n}f_0(n)\cdot f_{12}(n)-\tfrac{1}{2}\lambda_1 D_t\mathrm{e}^{-\frac{1}{2}D_n}f_1(n)\cdot f_2(n)-\tfrac{1}{2}\lambda_2 D_t\mathrm{e}^{\frac{1}{2}D_n}f_1(n)\cdot f_2(n)$$
$$+\lambda_1\mu_1\mathrm{e}^{-\frac{1}{2}D_n}f_1(n)\cdot f_2(n)-\lambda_2\mu_2\mathrm{e}^{\frac{1}{2}D_n}f_1(n)\cdot f_2(n)=0. \tag{11}$$

Therefore in order to prove proposition 2, it suffices to show that

$$D_z f_1\cdot f_{12}=[\lambda_2^{-1}D_t\mathrm{e}^{-D_n}+\lambda_2^{-1}\mu_2\mathrm{e}^{-D_n}+\gamma_2]f_1\cdot f_{12} \tag{12}$$
$$D_z f_2\cdot f_{12}=[\lambda_1^{-1}D_t\mathrm{e}^{-D_n}+\lambda_1^{-1}\mu_1\mathrm{e}^{-D_n}+\gamma_1]f_2\cdot f_{12} \tag{13}$$
$$[\mathrm{e}^{\frac{1}{2}D_n}D_t+\mu_2\mathrm{e}^{\frac{1}{2}D_n}-\lambda_2\mathrm{e}^{\frac{3}{2}D_n}+\omega_2\mathrm{e}^{-\frac{3}{2}D_n}]f_1\cdot f_{12}=0 \tag{14}$$
$$[\mathrm{e}^{\frac{1}{2}D_n}D_t+\mu_1\mathrm{e}^{\frac{1}{2}D_n}-\lambda_1\mathrm{e}^{\frac{3}{2}D_n}+\omega_1\mathrm{e}^{-\frac{3}{2}D_n}]f_2\cdot f_{12}=0. \tag{15}$$

Since f_1 and f_2 are two solutions of equation (4), then we have

$$[(D_t^2\mathrm{e}^{\frac{1}{2}D_n}-D_z\mathrm{e}^{\frac{1}{2}D_n})f_1\cdot f_1][\mathrm{e}^{\frac{1}{2}D_n}f_2\cdot f_2]-[\mathrm{e}^{\frac{1}{2}D_n}f_1\cdot f_1][(D_t^2\mathrm{e}^{\frac{1}{2}D_n}-D_z\mathrm{e}^{\frac{1}{2}D_n})f_2\cdot f_2]=0. \tag{16}$$

Using formulae (A3), (A4), (A11)–(A13), (9), (10) and $f_0 \overset{(\lambda_i,\mu_i,\gamma_i,\omega_i)}{\longrightarrow} f_i$ $(i=1,2)$, we can rewrite (16) as

$$\frac{1}{\lambda_1 c}\exp(\tfrac{1}{2}D_n)[(D_z-\lambda_2^{-1}D_t\mathrm{e}^{-D_n}-\lambda_2^{-1}\mu_2\mathrm{e}^{-D_n}-\gamma_2)f_1\cdot f_{12}]\cdot f_0f_2=0 \tag{17}$$

which implies that (12) holds. Similarly we can prove (13) also holds. Next, from

$$[(-D_z+\lambda_1^{-1}D_t\mathrm{e}^{-D_n}+\lambda_1^{-1}\mu_1\mathrm{e}^{-D_n}+\gamma_1)f_0\cdot f_1]f_2$$
$$-[(-D_z+\lambda_2^{-1}D_t\mathrm{e}^{-D_n}+\lambda_2^{-1}\mu_2\mathrm{e}^{-D_n}+\gamma_2)f_0\cdot f_2]f_1=0$$

we can deduce, by use of (A18), (9) and (11), that

$$D_z f_1(n)\cdot f_2(n)+(\gamma_1-\gamma_2)f_1(n)f_2(n)-\frac{1}{c\lambda_1\lambda_2}D_t\mathrm{e}^{-D_n}f_0(n)\cdot f_{12}(n)$$
$$-\frac{\mu_1+\mu_2}{c\lambda_1\lambda_2}\mathrm{e}^{-D_n}f_0(n)\cdot f_{12}(n)=0. \tag{18}$$

Finally, since f_1 and f_2 are two solutions of equation (5), we have that

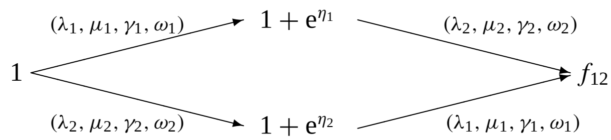
$$[(D_z e^{D_n} - D_t^2 e^{D_n} + 2e^{2D_n} - 2e^{D_n}) f_1 \cdot f_1][e^{D_n} f_2 \cdot f_2] \\ -[e^{D_n} f_1 \cdot f_1][(D_z e^{D_n} - D_t^2 e^{D_n} + 2e^{2D_n} - 2e^{D_n}) f_2 \cdot f_2] = 0. \tag{19}$$

Using formulae (A3), (A4), (A7), (A8), (A14)–(A17), (9), (10), (18) and $f_0 \overset{(\lambda_i, \mu_i, \gamma_i, \omega_i)}{\longrightarrow} f_i$ $(i = 1, 2)$, we can rewrite (19) as

$$-\frac{2}{c\lambda_1\lambda_2} \exp(-\tfrac{1}{2}D_n)[e^{-\frac{3}{2}D_n} f_0 \cdot f_2][e^{\frac{1}{2}D_n} D_t + \mu_2 e^{\frac{1}{2}D_n} - \lambda_2 e^{\frac{3}{2}D_n} + \omega_2 e^{-\frac{3}{2}D_n}) f_1 \cdot f_{12}] = 0 \tag{20}$$

which implies that (14) holds. Similarly we can prove that (15) also holds. Therefore we have completed the proof of proposition 2. □

As an application of the result, we can construct soliton solutions of the BC lattice. Choose for example $f_0 = 1$, $c = \frac{1}{\lambda_1 - \lambda_2}$. It is easily verified that



where

$$f_{12} = 1 + \frac{\lambda_1 e^{-p_1} - \lambda_2}{\lambda_1 - \lambda_2} e^{\eta_1} + \frac{\lambda_1 - \lambda_2 e^{-p_2}}{\lambda_1 - \lambda_2} e^{\eta_2} + \frac{\lambda_1 e^{-p_1} - \lambda_2 e^{-p_2}}{\lambda_1 - \lambda_2} e^{\eta_1 + \eta_2}$$

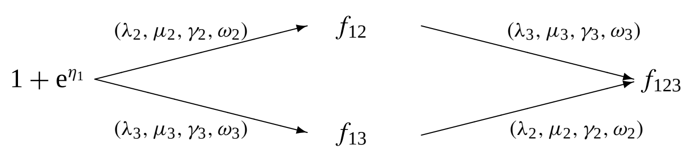
with

$$\eta_i = p_i n + q_i z + r_i t + \eta_i^0 \qquad q_i = \lambda_i^{-2}(e^{2p_i} - 1) \qquad r_i = \lambda_i^{-1}(e^{p_i} - 1)$$

and

$$\lambda_i = \pm\sqrt{\frac{e^{\frac{1}{2}p_i} - e^{\frac{3}{2}p_i}}{e^{\frac{3}{2}p_i} - e^{-\frac{3}{2}p_i}}} \qquad \mu_i = -\lambda_i^{-1} \qquad \gamma_i = \lambda_i^{-2} \qquad \omega_i = \lambda_i + \lambda_i^{-1}.$$

Thus f_{12} so obtained is a two-soliton solution of (4), (5). Furthermore, we have



where

$$f_{13} = 1 + \frac{\lambda_1 e^{-p_1} - \lambda_3}{\lambda_1 - \lambda_3} e^{\eta_1} + \frac{\lambda_1 - \lambda_3 e^{-p_3}}{\lambda_1 - \lambda_3} e^{\eta_3} + \frac{\lambda_1 e^{-p_1} - \lambda_3 e^{-p_3}}{\lambda_1 - \lambda_3} e^{\eta_1 + \eta_3}$$

and

$$f_{123} = 1 + K_1 e^{\eta_1} + K_2 e^{\eta_2} + K_3 e^{\eta_3} + K_{12} e^{\eta_1+\eta_2} + K_{13} e^{\eta_1+\eta_3} + K_{23} e^{\eta_2+\eta_3} + K_{123} e^{\eta_1+\eta_2+\eta_3}$$

with

$$K_i = \prod_{\substack{j=1\\ j\neq i}}^{3} \frac{\lambda_i e^{-p_i} - \lambda_j}{\lambda_i - \lambda_j}$$

$$K_{ij} = \frac{\lambda_i e^{-p_i} - \lambda_j e^{-p_j}}{\lambda_i - \lambda_j} \frac{\lambda_i e^{-p_i} - \lambda_{6-i-j}}{\lambda_i - \lambda_{6-i-j}} \frac{\lambda_j e^{-p_j} - \lambda_{6-i-j}}{\lambda_j - \lambda_{6-i-j}} \qquad i \neq j$$

$$K_{123} = \frac{\lambda_1 e^{-p_1} - \lambda_2 e^{-p_2}}{\lambda_1 - \lambda_2} \frac{\lambda_1 e^{-p_1} - \lambda_3 e^{-p_3}}{\lambda_1 - \lambda_3} \frac{\lambda_2 e^{-p_2} - \lambda_3 e^{-p_3}}{\lambda_2 - \lambda_3}.$$

Then f_{123} is a three-soliton solution of (4), (5) where

$$\eta_i = p_i n + q_i z + r_i t + \eta_i^0 \qquad q_i = \lambda_i^{-2}(e^{2p_i} - 1) \qquad r_i = \lambda_i^{-1}(e^{p_i} - 1)$$

and

$$\lambda_i = \pm\sqrt{\frac{e^{\frac{1}{2}p_i} - e^{\frac{3}{2}p_i}}{e^{\frac{3}{2}p_i} - e^{-\frac{3}{2}p_i}}} \qquad \mu_i = -\lambda_i^{-1} \qquad \gamma_i = \lambda_i^{-2} \qquad \omega_i = \lambda_i + \lambda_i^{-1}.$$

Generally, along this line, we can obtain multisoliton solutions for the BC lattice (4), (5) step by step.

4. Summary

The so-called BC lattice has been considered. By the dependent variable transformation, the BC lattice is transformed into a trilinear form. By introducing an auxiliary variable, we further transform it into the bilinear form. We have given a BT for the BC lattice as well as a nonlinear superposition formula for it. As an application of the obtained results, soliton solutions are derived.

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Appendix. Hirota bilinear operator identities

The following bilinear operator identities hold for arbitrary functions a, b, c and d

$$[D_z e^{\frac{1}{2}D_n} a\cdot a][e^{\frac{3}{2}D_n} a\cdot a] = -\tfrac{1}{2} D_z e^{\frac{1}{2}D_n}[e^{D_n} a\cdot a]\cdot[e^{D_n} a\cdot a] + \tfrac{1}{2} e^{\frac{1}{2}D_n}\{[D_z e^{D_n} a\cdot a]\cdot[e^{D_n} a\cdot a] + [e^{D_n} a\cdot a]\cdot[D_z e^{D_n} a\cdot a]\} \quad \text{(A1)}$$

$$e^{D_n}[D_z e^{\frac{1}{2}D_n} a\cdot a]\cdot[e^{-\frac{1}{2}D_n} a\cdot a] = \tfrac{1}{2} D_z e^{\frac{1}{2}D_n}[e^{D_n} a\cdot a]\cdot[e^{D_n} a\cdot a] + \tfrac{1}{2} e^{\frac{1}{2}D_n}\{[D_z e^{D_n} a\cdot a]\cdot[e^{D_n} a\cdot a] + [e^{D_n} a\cdot a]\cdot[D_z e^{-D_n} a\cdot a]\} \quad \text{(A2)}$$

$$[D_y e^{\delta D_n} a\cdot a][e^{\delta D_n} b\cdot b] - [D_y e^{\delta D_n} b\cdot b][e^{\delta D_n} a\cdot a] = 2\sinh(\delta D_n)(D_y a\cdot b)\cdot ab \quad \text{(A3)}$$

$$[D_t^2 e^{\delta D_n} a\cdot a][e^{\delta D_n} b\cdot b] - [D_t^2 e^{\delta D_n} b\cdot b][e^{\delta D_n} a\cdot a] = 2D_t\cosh(\delta D_n)(D_t a\cdot b)\cdot ab \quad \text{(A4)}$$

$$D_t\cosh(\tfrac{1}{2}D_n)(e^{-D_n} a\cdot b)\cdot ab = -\sinh(\tfrac{1}{2}D_n)\{[D_t e^{-D_n} a\cdot b]\cdot ab + [D_t a\cdot b]\cdot[e^{-D_n} a\cdot b]\} \quad \text{(A5)}$$

$$D_t \cosh(\delta D_n) a \cdot a = 0 \tag{A6}$$

$$\sinh(\delta D_n) a \cdot a = 0 \tag{A7}$$

$$\begin{aligned}[\mathrm{e}^{2D_n} a \cdot a][\mathrm{e}^{D_n} b \cdot b] - [\mathrm{e}^{D_n} a \cdot a][\mathrm{e}^{2D_n} b \cdot b] &= 2\sinh(\tfrac{1}{2}D_n)[\mathrm{e}^{\frac{3}{2}D_n} a \cdot b] \cdot [\mathrm{e}^{-\frac{3}{2}D_n} a \cdot b] \\ &= 2\sinh(\tfrac{3}{2}D_n)[\mathrm{e}^{\frac{1}{2}D_n} a \cdot b] \cdot [\mathrm{e}^{-\frac{1}{2}D_n} a \cdot b]\end{aligned} \tag{A8}$$

$$\begin{aligned}&D_t \cosh(D_n)(\mathrm{e}^{-D_n} a \cdot b) \cdot ab + \sinh(D_n)\{[D_t \mathrm{e}^{-D_n} a \cdot b] \cdot ab + [\mathrm{e}^{-D_n} a \cdot b] \cdot [D_t a \cdot b]\} \\ &\qquad = 2\sinh(\tfrac{1}{2}D_n)[\mathrm{e}^{-\frac{3}{2}D_n} a \cdot b] \cdot [D_t \mathrm{e}^{\frac{1}{2}D_n} a \cdot b]\end{aligned} \tag{A9}$$

$$\sinh(D_n)[\mathrm{e}^{-D_n} a \cdot b] \cdot ab = \sinh(\tfrac{1}{2}D_n)[\mathrm{e}^{-\frac{3}{2}D_n} a \cdot b][\mathrm{e}^{\frac{1}{2}D_n} a \cdot b] \tag{A10}$$

$$2\sinh(\delta D_n)(D_t a \cdot b) \cdot ab = D_t[\mathrm{e}^{\delta D_n} a \cdot b] \cdot [\mathrm{e}^{-\delta D_n} a \cdot b] \tag{A11}$$

$$D_z[\mathrm{e}^{\frac{1}{2}D_n} a \cdot b] \cdot [\mathrm{e}^{-\frac{1}{2}D_n} c \cdot d] = \mathrm{e}^{\frac{1}{2}D_n}[(D_z a \cdot d) \cdot cb - ad \cdot (D_z c \cdot b)] \tag{A12}$$

$$\begin{aligned}2D_t \cosh(\tfrac{1}{2}D_n)[\mathrm{e}^{-D_n} a \cdot b] \cdot cd &= \mathrm{e}^{\frac{1}{2}D_n}\{cb \cdot [D_t \mathrm{e}^{-D_n} a \cdot d] - [D_t \mathrm{e}^{-D_n} c \cdot b] \cdot ad \\ &\quad + [\mathrm{e}^{-D_n} c \cdot b] \cdot [D_t a \cdot d] - [D_t c \cdot b] \cdot [\mathrm{e}^{-D_n} a \cdot d]\}\end{aligned} \tag{A13}$$

$$\begin{aligned}&\sinh(D_n)\{[D_t \mathrm{e}^{-D_n} a \cdot b] \cdot cd + [\mathrm{e}^{-D_n} a \cdot b] \cdot [D_t c \cdot d]\} \\ &\qquad = \tfrac{1}{2}\mathrm{e}^{-\frac{1}{2}D_n}\{[D_t \mathrm{e}^{\frac{1}{2}D_n} a \cdot d] \cdot [\mathrm{e}^{-\frac{3}{2}D_n} c \cdot b] + [\mathrm{e}^{\frac{1}{2}D_n} a \cdot d] \cdot [D_t \mathrm{e}^{-\frac{3}{2}D_n} c \cdot b] \\ &\qquad\quad - [D_t \mathrm{e}^{-\frac{3}{2}D_n} a \cdot d] \cdot [\mathrm{e}^{\frac{1}{2}D_n} c \cdot b] - [\mathrm{e}^{-\frac{3}{2}D_n} a \cdot d] \cdot [D_t \mathrm{e}^{\frac{1}{2}D_n} c \cdot b]\}\end{aligned} \tag{A14}$$

$$\begin{aligned}2D_t \cosh(D_n)[\mathrm{e}^{-D_n} a \cdot b] \cdot cd &= \mathrm{e}^{-\frac{1}{2}D_n}\{[D_t \mathrm{e}^{\frac{1}{2}D_n} a \cdot d] \cdot [\mathrm{e}^{-\frac{3}{2}D_n} c \cdot b] \\ &\quad - [\mathrm{e}^{\frac{1}{2}D_n} a \cdot d] \cdot [D_t \mathrm{e}^{-\frac{3}{2}D_n} c \cdot b] + [D_t \mathrm{e}^{-\frac{3}{2}D_n} a \cdot d] \cdot [\mathrm{e}^{\frac{1}{2}D_n} c \cdot b] \\ &\quad - [\mathrm{e}^{-\frac{3}{2}D_n} a \cdot d] \cdot [D_t \mathrm{e}^{\frac{1}{2}D_n} c \cdot b]\}\end{aligned} \tag{A15}$$

$$\sinh(D_n)(\mathrm{e}^{-D_n} a \cdot b) \cdot cd = \tfrac{1}{2}\mathrm{e}^{-\frac{1}{2}D_n}\{[\mathrm{e}^{\frac{1}{2}D_n} a \cdot d] \cdot [\mathrm{e}^{-\frac{3}{2}D_n} c \cdot b] - [\mathrm{e}^{-\frac{3}{2}D_n} a \cdot d] \cdot [\mathrm{e}^{\frac{1}{2}D_n} c \cdot b]\} \tag{A16}$$

$$\begin{aligned}\sinh(\tfrac{3}{2}D_n)[\mathrm{e}^{-\frac{1}{2}D_n} a \cdot b] \cdot [\mathrm{e}^{\frac{1}{2}D_n} c \cdot d] &= \tfrac{1}{2}\mathrm{e}^{-\frac{1}{2}D_n}\{[\mathrm{e}^{\frac{3}{2}D_n} a \cdot d] \cdot [\mathrm{e}^{-\frac{3}{2}D_n} c \cdot b] \\ &\quad - [\mathrm{e}^{-\frac{3}{2}D_n} a \cdot d] \cdot [\mathrm{e}^{\frac{3}{2}D_n} c \cdot b]\}\end{aligned} \tag{A17}$$

$$(D_z a \cdot b)c - (D_t a \cdot c)b = -a D_z b \cdot c. \tag{A18}$$

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